Journal of Basic and Applied Engineering Research
p-ISSN: 2350-0077, e-ISSN: 2350-0255; Volume-7, Issue-1; October-December, 2020 pp. 47-51

http://www.krishisanskriti.org/Publication.html

L^r Inequalities for Polynomials with Restricted Zeros

Barchand Chanam

Associate Professor, National Institute of Technology Manipur, Manipur, India
barchand_2004@yahoo.co.in

Abstract: If $p(z)$ is a polynomial of degree n which does not vanish in $|z|<k$, where $k\ge 1$, then for each $r>0$ and $1\le s<n$, Aziz and Rather [J. Math. Anal. Appl., 289(2004), 14-29] proved

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p^s\left(e^{i\theta}\right)\right|^r d\theta\right)^{\frac{1}{r}} \le \frac{n(n-1)......(n-s+1)}{\left(\frac{1}{2\pi}\int_0^{2\pi}\left|k^s+e^{i\alpha}\right|^r d\alpha\right)^{\frac{1}{r}}} \times\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta\right)^{\frac{1}{r}}.$$

In this paper, we prove an improvement of this inequality which besides gives some interesting results as corollaries, includes some well-known results as special cases.

1. INTRODUCTION

Let $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ be a polynomial of degree n and $p'(z)$ be its derivative, then for $r>0$,

$$\left\{\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le n\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}. \qquad (1.1)$$

If we let $r\to\infty$ in (1.1) and make use of the well-known fact from analysis [13, 14] that

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} = \max_{|z|=1}\left|p(z)\right|, \qquad (1.2)$$

we obtain the following inequalities

$$\max_{|z|=1}\left|p'(z)\right| \le n\max_{|z|=1}\left|p(z)\right|, \qquad (1.3)$$

Inequality (1.3) is a classical result due to Bernstein [3].

If we restrict ourselves to the class of polynomials having no zero in $|z|<1$, then inequality (1.1) can be improved. In fact, the following results are known.

Theorem A. *If is a polynomial of degree n having no zero in $|z|<1$, then for each $r>0$,*

$$\left\{\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le nC_r\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}, \qquad (1.4)$$

where

$$C_r=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{i\alpha}\right|^r d\alpha\right\}^{-\frac{1}{r}}.$$

In (1.4), *equality occurs for* $p(z)=\alpha z^n+\beta$, $|\alpha|=|\beta|$.

For $r\ge 1$, Theorem A was found by de-Bruijn [4] and later independently proved by Rahman [10]. For the special case $r=2$, it was proved by Lax [9]. Rahman and Schmeisser [11] showed that (1.4) remain valid for $0<r<1$ as well.

For the class of polynomials having no zero in the disc $|z|<k$, $k\ge 1$, Govil and Rahman [7] proved the following inequality (1.5) for $r\ge 1$.

Later it was shown by Gardner and Weems [6], and independently by Rather [12] that inequality (1.5) also holds for $0<r<1$.

Theorem B. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\ge 1$, then for $r>0$,*

$$\left\{\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \le nF_r\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}, \qquad (1.5)$$

where

$$F_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| k + e^{i\alpha} \right|^r d\alpha \right\}^{-\frac{1}{r}}.$$

For the same class of polynomials, Aziz and Shah [2] considered the sth derivative of $p(z), 1 \le s < n$, and generalized inequality (1.5) by proving

Theorem C. *If $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, where $k \ge 1$, then for each $r > 0$ and $1 \le s < n$,*

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \left| p^s\left(e^{i\theta}\right) \right|^r d\theta \right)^{\frac{1}{r}} \le \frac{n(n-1)......(n-s+1)}{\left(\frac{1}{2\pi} \int_0^{2\pi} \left| k^s + e^{i\alpha} \right|^r d\alpha \right)^{\frac{1}{r}}} \times \left(\frac{1}{2\pi} \int_0^{2\pi} \left| p\left(e^{i\theta}\right) \right|^r d\theta \right)^{\frac{1}{r}} \tag{1.6}$$

By involving some coefficients and $m = \min_{|z|=k} |p(z)|$, we present a generalization and an improvement of Theorem C. More precisely, we obtain

Theorem. *If $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, where $k \ge 1$, then for each $r > 0$, $1 \le s < n$, and for every real or complex number β such that $|\beta| < \frac{1}{k^n}$,*

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \left| p^s\left(e^{i\theta}\right) + \beta mn(n-1)....(n-s+1) e^{i\theta(n-s)} \right|^r d\theta \right)^{\frac{1}{r}} \le \frac{n(n-1).......(n-s+1)}{\left(\frac{1}{2\pi} \int_0^{2\pi} \left| \delta_{k,s} + e^{i\alpha} \right|^r d\alpha \right)^{\frac{1}{r}}} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| p\left(e^{i\theta}\right) + \beta m e^{i\theta n} \right|^r d\theta \right)^{\frac{1}{r}} \tag{1.7}$$

where

$$m = \min_{|z|=k} |p(z)|$$

and

$$\delta_{k,s} = \left\{ \frac{c(n,s)|a_0| k^{s+1} + |a_s| k^{2s}}{c(n,s)|a_0| + |a_s| k^{s+1}} \right\} \text{with } c(n,s) = \frac{n!}{s!(n-s)!}.$$

Remark 1. If we put $\beta = 0$, our theorem directly reduces to the following result proved by Aziz and Rather [1] and is an improvement of inequality (1.6) due to Aziz and Shah [2].

Corollary 1. *If $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, where $k \ge 1$, then for each $r > 0$ and $1 \le s < n$,*

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \left| p^s\left(e^{i\theta}\right) \right|^r d\theta \right)^{\frac{1}{r}} \le \frac{n(n-1).......(n-s+1)}{\left(\frac{1}{2\pi} \int_0^{2\pi} \left| \delta_{k,s} + e^{i\alpha} \right|^r d\alpha \right)^{\frac{1}{r}}} \times \left(\frac{1}{2\pi} \int_0^{2\pi} \left| p\left(e^{i\theta}\right) \right|^r d\theta \right)^{\frac{1}{r}}, \tag{1.8}$$

where

$$\delta_{k,s} = \left\{ \frac{c(n,s)|a_0| k^{s+1} + |a_s| k^{2s}}{c(n,s)|a_0| + |a_s| k^{s+1}} \right\} \text{with } c(n,s) = \frac{n!}{s!(n-s)!}.$$

Remark 2. Taking limit of both sides of (1.7) as $r \to \infty$, we have

$$\begin{aligned} \max_{|z|=1} \left| p^s(z) + \beta mn(n-1).....(n-s+1) z^{n-s} \right| &\le \frac{n(n-1)......(n-s+1)}{\delta_{k,s}+1} \times \max_{|z|=1} \left| p(z) + \beta m z^n \right| \\ &\le \frac{n(n-1)........(n-s+1)}{\delta_{k,s}+1} \times \left\{ \max_{|z|=1} |p(z)| + |\beta| m \right\}. \end{aligned} \tag{1.9}$$

If we choose z_0 such that $\max_{|z|=1} \left| p^s(z) \right| = \left| p^s(z_0) \right|$. It is clear that

$$\left| p^s(z_0) + \beta mn(n-1).......(n-s+1) z_0^{n-s} \right| \le \max_{|z|=1} \left| p^s(z) + \beta mn(n-1)......(n-s+1) z^{n-s} \right|,$$

and hence (1.9), in particular, gives

$$\left| p^s(z_0) + \beta mn(n-1).....(n-s+1) z_0^{n-s} \right| \le \frac{n(n-1)............(n-s+1)}{\delta_{k,s}+1} \left\{ \max_{|z|=1} |p(z)| + |\beta| m \right\}. \tag{1.10}$$

Now choosing the argument of β in (1.10) suitably such that

$$\left|p^{s}(z_0)+\beta mn(n-1)......(n-s+1)z_0^{n-s}\right|$$
$$=\left|p^{s}(z_0)\right|+|\beta|mn(n-1)......(n-s+1).$$

Then inequality (1.10) becomes

$$\left|p^{s}(z_0)\right|\le\frac{n(n-1).....(n-s+1)}{\delta_{k,s}+1}$$
$$\times\left\{\max_{|z|=1}|p(z)|+|\beta|m\right\}-|\beta|mn(n-1)...(n-s+1),$$

which is equivalent to

$$\max_{|z|=1}\left|p^{s}(z)\right|\le\frac{n(n-1).......(n-s+1)}{\delta_{k,s}+1}\max_{|z|=1}|p(z)|$$
$$-\left\{1-\frac{1}{\delta_{k,s}}\right\}n(n-1)..........(n-s+1)|\beta|m.$$

Finally, letting $|\beta|\to\frac{1}{k^n}$, we obtain

$$\max_{|z|=1}\left|p^{s}(z)\right|\le\left\{\frac{c(n,s)|a_0|+|a_s|k^{s+1}}{c(n,s)|a_0|\left(1+k^{s+1}\right)+|a_s|\left(k^{s+1}+k^{2s}\right)}\right\}$$
$$\times n(n-1)....(n-s+1)\max_{|z|=1}|p(z)|$$
$$-\left\{\frac{c(n,s)|a_0|k^{s+1}+|a_s|k^{2s}}{c(n,s)|a_0|\left(1+k^{s+1}\right)+|a_s|\left(k^{s+1}+k^{2s}\right)}\right\}$$
$$\times n(n-1)......(n-s+1)\frac{m}{k^n}. \quad (1.11)$$

Remark 3. Inequality (1.11) improves upon the following result due to Aziz and Rather [1].

Corollary 2. *If* $p(z)=\sum_{\upsilon=0}^{n}a_\upsilon z^\upsilon$ *is a polynomial of degree* n *which does not vanish in* $|z|<k$, *where* $k\ge1$, *then*

$$\max_{|z|=1}\left|p^{s}(z)\right|\le\left\{\frac{c(n,s)|a_0|+|a_s|k^{s+1}}{c(n,s)|a_0|\left(1+k^{s+1}\right)+|a_s|\left(k^{s+1}+k^{2s}\right)}\right\}$$
$$\times\max_{|z|=1}|p(z)|. \quad (1.12)$$

Remark 4. For $s=1$, (1.12) becomes

$$\max_{|z|=1}|p'(z)|\le\left\{\frac{n|a_0|+|a_1|k^2}{n|a_0|\left(1+k^2\right)+2|a_1|k^2}\right\}n\max_{|z|=1}|p(z)|$$
$$-\left\{\frac{n|a_0|k^2+|a_1|k^2}{n|a_0|\left(1+k^2\right)+2|a_1|k^2}\right\}n\frac{m}{k^n}.$$

which gives an improvement of the following result due to Govil et al. [8].

Corollary 3. *If* $p(z)=\sum_{\upsilon=0}^{n}a_\upsilon z^\upsilon$ *is a polynomial of degree* n *which does not vanish in* $|z|<k$, *where* $k\ge1$, *then*

$$\max_{|z|=1}|p'(z)|\le\left\{\frac{n|a_0|+|a_1|k^2}{n|a_0|\left(1+k^2\right)+2|a_1|k^2}\right\}n\max_{|z|=1}|p(z)|.$$

2. Lemma

For the proof of the theorem, we require the following lemmas.

Lemma 2.1. *If* $p(z)$ *is a polynomial of degree* n *and* $q(z)=z^n\,\overline{p\left(\frac{1}{\bar z}\right)}$, *then for each* α, $0\le\alpha<2\pi$ *and* $r>0$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|q'\left(e^{i\theta}\right)+e^{i\alpha}p'\left(e^{i\theta}\right)\right|^r d\theta\,d\alpha\le2\pi n^r\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta. \quad (2.1)$$

The above lemma is due to Aziz and Rather [1].

Lemma 2.2. *Let z be complex and independent of* α, *where* α *is real, then for* $r>0$,

$$\int_0^{2\pi}\left|1+ze^{i\alpha}\right|^r d\alpha=\int_0^{2\pi}\left|e^{i\alpha}+|z|\right|^r d\alpha. \quad (2.2)$$

This lemma is due to Govil [5].

Lemma 2.3. *If* $p(z)$ *is a polynomial of degree n which does not vanish in* $|z|<k$, *where* $k\ge1$, *then for* $1\le s<n$, *and* $|z|=1$,

$$\delta_{k,s}\left|p^{(s)}(z)\right|\le\left|q^{(s)}(z)\right|, \quad (2.3)$$

where

$$\delta_{k,s} = \left\{ \frac{c(n,s)|a_0|k^{s+1} + |a_s|k^{2s}}{c(n,s)|a_0| + |a_s|k^{s+1}} \right\}$$

with $c(n,s) = \dfrac{n!}{s!(n-s)!}$.

Proof of the Theorem

Since β, a real or complex number such that $|\beta| < \dfrac{1}{k^n}$, therefore on $|z| = k$,

$$|m\beta z^n| = m|\beta|k^n < m = \min_{|z|=k} |p(z)| .$$

By Rouché's theorem, the polynomial $P(z) = p(z) + m\beta z^n$ will have no zero in $|z| < k$, $k \geq 1$. Further, the case for $m = 0$ is trivially true.

Let $F(z) = Q(z) + e^{i\alpha} P(z),\ \alpha \in \Re$ where $Q(z) = z^n \overline{P\left(\frac{1}{\bar z}\right)}$ the reciprocal polynomial is. Then $F(z)$ is a polynomial of degree n and $F^{(s)}(z) = Q^{(s)}(z) + e^{i\alpha} Q^{(s)}(z)$ is a polynomial of degree $n-s$. By repeated application of inequality (1.1), it follows for each $r > 0$,

$$\begin{aligned}\int_0^{2\pi} \left|Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta})\right|^r d\theta &\leq (n-s+1)^r \\ &\quad \times \int_0^{2\pi} \left|Q^{(s-1)}(e^{i\theta}) + e^{i\alpha} P^{(s-1)}(e^{i\theta})\right|^r d\theta \\ &\leq (n-s+1)^r (n-s+2)^r \ldots (n-1)^r \\ &\quad \times \int_0^{2\pi} \left|Q'(e^{i\theta}) + e^{i\alpha} P'(e^{i\theta})\right|^r d\theta . \end{aligned} \tag{3.1}$$

Integrating (3.1) with respect to α on $[0, 2\pi)$, and using lemma 2.1, we get

$$\begin{aligned}\int_0^{2\pi} \left|Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta})\right|^r d\theta &\leq (n-s+1)^r (n-s+2)^r \ldots \\ &\quad \times (n-1)^r \int_0^{2\pi}\int_0^{2\pi} \left|Q'(e^{i\theta}) + e^{i\alpha} P'(e^{i\theta})\right|^r d\theta\, d\alpha \\ &\leq (n-s+1)^r (n-s+2)^r \ldots (n-1)^r n^r \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta .\end{aligned}$$

(3.2)

By lemma 2.3, we have for $1 \leq s < n$, for $|z| = 1$,

$$\delta_{k,s} \left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z)\right| , \tag{3.3}$$

where $\delta_{k,s} \geq 1$ is as defined in lemma 2.3. It can be easily verified that for every real number α and $R \geq R' \geq 1$, $|R + e^{i\alpha}| \geq |R' + e^{i\alpha}|$.

This implies for each $r > 0$,

$$\int_0^{2\pi} \left| R + e^{i\alpha}\right|^r d\alpha \geq \int_0^{2\pi} \left| R' + e^{i\alpha}\right|^r d\alpha .$$

For points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $P^{(s)}(e^{i\theta}) \neq 0$, we take

$R = \dfrac{\left|Q^{(s)}(e^{i\theta})\right|}{\left|P^{(s)}(e^{i\theta})\right|}$ and $R' = \delta_{k,s}$, then by (3.3), $R \geq R' \geq 1$,

$$\begin{aligned}\int_0^{2\pi} \left|Q^{(s)}(e^{i\theta}) + e^{i\alpha} P^{(s)}(e^{i\theta})\right|^r d\alpha &= \left|P^{(s)}(e^{i\theta})\right|^r \int_0^{2\pi} \left|\frac{Q^{(s)}(e^{i\theta})}{P^{(s)}(e^{i\theta})} + e^{i\alpha}\right|^r d\alpha \\ &\geq \left|P^{(s)}(e^{i\theta})\right|^r \int_0^{2\pi} \left|\delta_{k,s} + e^{i\alpha}\right|^r d\alpha \end{aligned} \tag{3.4}$$

Also for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $P^{(s)}(e^{i\theta}) = 0$, inequality (3.4) follows trivially. Using (3.2) in (3.4), it is concluded that for each $r > 0$,

$$\begin{aligned}\int_0^{2\pi} \left|\delta_{k,s} + e^{i\alpha}\right|^r d\alpha \int_0^{2\pi} \left|P^{(s)}(e^{i\theta})\right|^r d\theta &\leq (n-s+1)^r (n-s+2)^r \ldots \\ &\quad \times (n-1)^r n^r 2\pi \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta \end{aligned}$$

. (3.5)

On replacing $P(z)$ in (3.5) by $p(z) + m\beta z^n$, we have

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \left|p^s(e^{i\theta}) + \beta mn(n-1)\ldots\ldots\ldots..(n-s+1)e^{i(n-s)\theta}\right|^r d\theta\right)^{\frac{1}{r}}$$

$$\leq \frac{n(n-1)\ldots\ldots..(n-s+1)}{\left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left|\delta_{k,s} + e^{i\alpha}\right|^r d\alpha\right)^{\frac{1}{r}}} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|p(e^{i\theta}) + \beta m e^{i\theta n}\right|^r d\theta\right)^{\frac{1}{r}}$$

This proves the Theorem.

REFERENCES

[1] Aziz A. and Rather N.A., ‘‘Some Zygmund type L^q inequalities for polynomials,’’ *J. Math. Anal. Appl.,* 289, pp. 14-29, 2004.

[2] Aziz A. and Shah W. M., ‘‘ L^p inequalities for polynomials with restricted zeros,’’ Proc. Indian Acad. Sci. Math. Sci. 108, pp. 63-68, 1998.

[3] S. Bernstein, ‘‘Lecons Sur Les Propriétés extrémales et la meilleure approximation des functions analytiques d’une fonctions reele,’’ Paris, 1926.

[4] de-Bruijn N.G., ‘‘Inequalities concerning polynomials in the complex domain,’’ *Nederl. Akad. Wetench. Proc. Ser. A,* 50, pp.1265-1272, 1947, *Indag. Math.,* 9, pp. 591-598, 1947.

[5] Gardner R.B. and Govil N.K., ‘‘An L^p inequality for a polynomial and its derivative,’’ *J. Math. Anal. Appl.,* 194, pp. 720-726, 1995.

[6] Gardner R.B. and Weems A., ‘‘A Bernstein-type of L^p inequality for a certain class of polynomials,’’ *J. Math. Anal. Appl.,* 219, pp. 472-478, 1998,.

[7] Govil N.K. and Rahman Q.I., ‘‘Functions of exponential type not vanishing in a half-plane and related polynomials,’’ *Trans. Amer. Math. Soc.,* 137, pp. 501-517, 1969.

[8] Govil N.K., Rahman Q.I. and Schmeisser G., ‘‘On the derivative of a polynomial,’’ *Illinois J. Math.,* 23, 319-329, 1979.

[9] Lax P.D., ‘‘Proof of a conjecture of P. Erdös on the derivative of a polynomial,’’ *Bull. Amer. Math. Soc.,* 50, pp. 509-513, 1944.

[10] Rahman Q.I., ‘‘Functions of exponential type,’’ *Trans. Amer. Math. Soc.,* 135, pp. 295-309, 1969.

[11] Rahman Q.I. and Schmeisser G., ‘‘ L^p inequalities for polynomials,’’ *J. Approx. Theory,* 53, pp. 26-32, 1988.

[12] Rather N.A., ‘‘Extremal properties and location of the zeros of polynomials,’’ *Ph.D. Thesis,* University of Kashmir, 1998.

[13] Rudin W., ‘’Real and complex Analysis,’’ *Tata Mcgraw-Hill Publishing Company (Reprinted in India),* 1977.

[14] Taylor A.E., ‘‘Introduction to Functional Analysis,’’ *John Wiley and Sons, Inc. New York,* 1958.